

Lower bounds for pseudodifferential operators with a radial symbol.

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ABSTRACT. In this paper we establish explicit lower bounds for pseudodifferential operators with a radial symbol. The proofs use classical Weyl calculus techniques and some useful, if not celebrated, properties of the Laguerre polynomials.

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1. Introduction.

If a function F defined on \mathbb{R}^{2d} is smooth and has bounded derivatives, the Weyl calculus associates with it a pseudodifferential operator $Op_h^{Weyl}(F)$ which is bounded on $L^2(\mathbb{R}^d)$ and satisfies, for all f and g in $\mathcal{S}(\mathbb{R}^d)$,

$$(1.1) \quad \langle Op_h^{Weyl}(F)f, g \rangle = (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} F(Z) H_h(f, g, Z) dZ,$$

where $H_h(f, g, \cdot)$ is the Wigner function

$$(1.2) \quad H_h(f, g, Z) = \int_{\mathbb{R}^d} e^{-\frac{it}{h} \cdot \zeta} f\left(z + \frac{t}{2}\right) \overline{g\left(z - \frac{t}{2}\right)} dt \quad Z = (z, \zeta) \in \mathbb{R}^{2d}.$$

For this form of the definition, see [U], [L] or [C-R], Chapter II, Proposition 14.

The different variants of Gårding's inequality prove that, if $F \geq 0$, the operator $Op_h^{Weyl}(F)$ is roughly ≥ 0 . More precisely, according to the classical Gårding's inequality (see [HO] or [L]), the non negativity of F implies the existence of a positive constant C , independent of h , such that, for all sufficiently small h and for all f in $\mathcal{S}(\mathbb{R}^d)$:

$$(1.3) \quad \langle Op_h^{Weyl}(F)f, f \rangle \geq -Ch \|f\|_{L^2(\mathbb{R}^d)}^2.$$

See [L-N] for other similar results. This inequality holds for systems of operators, whereas the more precise Fefferman-Phong inequality [F-P] is valid only for scalar operators. Fefferman-Phong's inequality states that, under the same hypotheses as Gårding's inequality, one has, for all h in $(0, 1)$ and all f in $\mathcal{S}(\mathbb{R}^d)$:

$$(1.4) \quad \langle Op_h^{Weyl}(F)f, f \rangle \geq -Ch^2 \|f\|_{L^2(\mathbb{R}^d)}^2.$$

See [MAR] for these semiclassical versions. Sometimes the non negativity of F implies the exact non negativity of the operator, for example in the simple case when F depends on x or on ξ only. It is possible, too, to apply Melin's inequality. To take only one example, let $F \geq 0$ attain its minimum only once, for a nondegenerate critical point. In this case (and in other analogous situations), Melin's inequality ensures the exact non negativity of $Op_h^{Weyl}(F)$ for a sufficiently small h . See [B-N] or [L-L] for cases when the difference between $F(x, \xi)$ and its minimum is equivalent to a power, greater than 2, of the distance between (x, ξ) and the unique point where the minimum is attained.

In this article we are interested in the case when F is radial. We assume that there exists a function Φ defined on \mathbb{R} such that

$$(1.5) \quad F(x, \xi) = \Phi(|x|^2 + |\xi|^2) \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Moreover, we suppose that Φ is nondecreasing on $[0, \infty)$ and such that F is smooth, with bounded derivatives. In this case, we aim at giving an explicit lower bound on the spectrum of the operator $Op_h^{Weyl}(F)$. The main result of this paper is the following theorem.

Theorem 1.1 *Let F be a smooth function defined on \mathbb{R}^{2d} , bounded as well as all its derivatives. Assume that F is of the form (1.5), where Φ is a non decreasing function defined on $[0, \infty)$. Then for all f in $\mathcal{S}(\mathbb{R}^d)$,*

$$(1.6) \quad \langle Op_h^{Weyl}(F)f, f \rangle \geq \frac{1}{h} \int_0^\infty \Phi(t) e^{-\frac{t}{h}} dt \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Remarks

1 - We do not need to assume that $\Phi \geq 0$ to ensure the non negativity of the operator. The non negativity of the integral suffices.

2 - In the case when Φ is not flat at the origin, let $m \geq 1$ be the smallest integer for which $\Phi^{(m)}(0) \neq 0$. Then one can see that

$$\frac{1}{h} \int_0^\infty \Phi(t) e^{-\frac{t}{h}} dt = \Phi(0) + \Phi^{(m)}(0) h^m + \mathcal{O}(h^{m+1}).$$

3 - The result can be applied to symbols F depending on the distance from another point (x_0, ξ_0) for, if $\tau F(x, \xi) = F(x + x_0, \xi + \xi_0)$ and $Tf(u) = e^{i(\xi_0/h)(u-x_0)} f(u - x_0)$, then

$$\langle Op_h^{Weyl}(\tau F)f, g \rangle = \langle Op_h^{Weyl}(F)Tf, Tg \rangle.$$

We are greatly indebted to N. Lerner for the reference [A-G].

2. Proof of Theorem 1.1.

We denote by $(H_n)_{(n \geq 0)}$ the sequence of the Hermite functions. It is a Hermitian basis of $L^2(\mathbb{R})$, satisfying

$$(2.1) \quad (D^2 + x^2)H_n = (2n + 1)H_n.$$

For each multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, we set :

$$(2.2) \quad u_\alpha(x) = \prod_{j=1}^d H_{\alpha_j}(x_j).$$

These functions form a Hermitian basis of $L^2(\mathbb{R}^d)$.

We shall need the Laguerre polynomials as well, which are defined by

$$(2.3) \quad L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}).$$

One has :

$$(2.4) \quad L_0(x) = 1 \quad L_1(x) = 1 - x \quad L_2(x) = \frac{x^2}{2} - 2x + 1.$$

Theorem 1.1 is a consequence of the following proposition, in which the parameter h is equal to 1 and the Weyl operator $Op_1^{Weyl}(F)$ is denoted by $Op^{Weyl}(F)$.

Proposition 2.1 *Under the hypotheses of Theorem 1.1 one has, for all multi-indices α and β such that $\alpha \neq \beta$:*

$$(2.5) \quad \langle Op^{Weyl}(F)u_\alpha, u_\beta \rangle = 0.$$

For each multi-index α :

$$(2.6) \quad \langle Op^{Weyl}(F)u_\alpha, u_\alpha \rangle = 2^{-d} \left[\Phi(0)V_\alpha(0) + \frac{1}{2} \int_0^\infty \Phi'(t/2)V_\alpha(t)dt \right],$$

with

$$(2.7) \quad V_\alpha(X) = 4e^{-\frac{X}{2}} \sum_{k=0}^{d-1} C_{d-1}^k T_{|\alpha|+k}(X),$$

where we set, for all integer n ,

$$(2.8) \quad T_n(X) = \left[\sum_{k=0}^{n-1} (-1)^k L_k(X) \right] + \frac{(-1)^n}{2} L_n(X).$$

Proof of (2.5). Let α and β be two different multi-indices and let $j \leq d$ be such that $\alpha_j \neq \beta_j$. Set $P_j = D_j^2 + x_j^2$. According to (2.1) we have :

$$2(\alpha_j - \beta_j) \langle Op^{Weyl}(F)u_\alpha, u_\beta \rangle = \langle Op^{Weyl}(F)P_j u_\alpha, u_\beta \rangle - \langle Op^{Weyl}(F)u_\alpha, P_j u_\beta \rangle.$$

The fact that F is radial implies that $x_j \frac{\partial F}{\partial \xi_j} - \xi_j \frac{\partial F}{\partial x_j} = 0$ which, in turn, implies that $Op^{Weyl}(F)$ and P_j commute, thanks to properties of the Weyl calculus. Consequently, the right term of the above inequality is equal to 0, which proves (2.5).

Proof of (2.6). For each multi-index α , the Wigner function $H(u_\alpha, u_\alpha)$ (where the parameter h , equal to 1, is omitted), satisfies:

$$(2.9) \quad H(u_\alpha, u_\alpha)(x, \xi) = 2^d (-1)^{|\alpha|} e^{-(|x|^2 + |\xi|^2)} \prod_{j=1}^d L_{\alpha_j}(2(x_j^2 + \xi_j^2)).$$

See, for example, [FO] or [J-L-V]. Hence, if F is as in Theorem 1.1,

$$\langle Op^{Weyl}(F)u_\alpha, u_\alpha \rangle = (2\pi)^{-d} 2^d (-1)^{|\alpha|} \int_{\mathbb{R}^{2d}} \Phi(|x|^2 + |\xi|^2) e^{-(|x|^2 + |\xi|^2)} \prod_{j=1}^d L_{\alpha_j}(2(x_j^2 + \xi_j^2)) dx d\xi.$$

The change of variables $t_j = 2(x_j^2 + \xi_j^2)$ allows to write :

$$\langle Op^{Weyl}(F)u_\alpha, u_\alpha \rangle = (2\pi)^{-d} 2^d (-1)^{|\alpha|} (\pi/2)^d \int_{[0, \infty)^d} \Phi((t_1 + \dots + t_d)/2) e^{-\frac{1}{2}(t_1 + \dots + t_d)} \prod_{j=1}^d L_{\alpha_j}(t_j) dt_1 \dots dt_d.$$

This equality can be written as

$$\langle Op^{Weyl}(F)u_\alpha, u_\alpha \rangle = (2\pi)^{-d} 2^d (\pi/2)^d \int_0^\infty \Phi(X/2) U_\alpha(X) dX,$$

with :

$$U_\alpha(X) = (-1)^{|\alpha|} e^{-\frac{X}{2}} \int_{\Omega_d(X)} L_{\alpha_d}(X - t_1 - \dots - t_{d-1}) \prod_{j=1}^{d-1} L_{\alpha_j}(t_j) dt_1 \dots dt_{d-1},$$

where

$$\Omega_d(X) = \{(t_1, \dots, t_{d-1}), \quad t_j > 0, \quad t_1 + \dots + t_{d-1} < X\}.$$

The equality (2.6) will be a consequence of an integration by parts using the following lemma.

Lemma 2.2 *We have:*

$$(2.10) \quad U_\alpha(X) = -V'_\alpha(X)$$

where V_α is defined by (2.7) and (2.8).

Proof of Lemma 2.2. One knows (cf [M-O-S], section 5.5.2) that

$$(2.11) \quad \int_0^X L_{\alpha_1}(t) L_{\alpha_2}(X-t) dt = L_{\alpha_1+\alpha_2}(X) - L_{\alpha_1+\alpha_2+1}(X).$$

It follows, by induction on d , that

$$\int_{\Omega_d(X)} L_{\alpha_d}(X - t_1 - \dots - t_{d-1}) \prod_{j=1}^{d-1} L_{\alpha_j}(t_j) dt_1 \dots dt_{d-1} = \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k L_{|\alpha|+k}(X).$$

Hence

$$U_\alpha(X) = e^{-\frac{X}{2}} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^{|\alpha|+k} L_{|\alpha|+k}(X).$$

Using the recurrence relation $L'_{k+1}(t) = L'_k(t) - L_k(t)$, we prove (for example by induction) that for all integer n :

$$\frac{d}{dt} e^{-\frac{t}{2}} T_n(t) = \frac{(-1)^{n+1}}{4} L_n(t) e^{-\frac{t}{2}}.$$

The equality (2.10) of the Lemma follows from (2.7) and from the above identities.

End of the proof of Theorem 1.1. We shall begin by proving (1.6) for $h = 1$. Set

$$(2.12) \quad S_n(X) = \sum_{k=0}^n (-1)^k L_k(X).$$

Using the recurrence relation $L'_{k+1}(t) = L'_k(t) - L_k(t)$, one verifies, by induction, that for all n :

$$T'_n(X) = \frac{1}{2} S_{n-1}(X).$$

Since $L_n(0) = 1$ for all n , we see that $T_n(0) = 1/2$ and that

$$T_n(X) = \frac{1}{2} + \frac{1}{2} \int_0^X S_{n-1}(t) dt.$$

According to [A-G], Theorem 12 (see [F] as well), $S_n(X) \geq 0$ for all $n \geq 0$ and for all $X \geq 0$. Therefore $T_n(X) \geq 1/2$ for all n and X , and, using (2.7):

$$(2.13) \quad V_\alpha(X) \geq 2^d e^{-\frac{X}{2}}.$$

Since $T_n(0) = 1/2$, $V_\alpha(0) = 2^d$. Hence, if $\Phi' \geq 0$, one gets :

$$(2.14) \quad \Phi(0)V_\alpha(0) + \frac{1}{2} \int_0^\infty \Phi'(t/2)V_\alpha(t)dt \geq 2^d \int_0^\infty \Phi(t)e^{-t}dt.$$

The inequality (1.6), for $h = 1$, follows from (2.5), (2.6) and (2.14). For an arbitrary $h > 0$, it suffices to apply the above result to the function $F_h(x, \xi) = F(h^{1/2}x, h^{1/2}\xi)$, that is to say, to the function $\Phi_h(t) = \Phi(th)$.

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